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Probability Theory and Applications (MA208)
Problem Sheet - 10

The Moment-Generating Function

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1. Suppose that X has pdf given by
$$f(x) = 2x, \quad 0 \leq x \leq 1.$$
   (a) Determine the mgf of X.
   (b) Using the mgf, evaluate $E(X)$ and $V(X)$ and check your answer . (See Note, p.206.)

2. (a) Find the mgf of the voltage (including noise) as discussed in Problem 7.25.
   (b) Using the mgf, obtain the expected value and variance of this voltage .

3. Suppose that X has the following pdf:
$$f(x) = \lambda e^{-\lambda(x-a)}, \quad x \geq a.$$
(This is known as a two-parameter exponential distribution.)
   (a) Find the mgf of X.
   (b) Using the mgf, find $E(X)$ and $V(X)$.

4. Let X be the outcome when a fair die is tossed.
   (a) Find the mgf of X.
   (b) Using the mgf, find $E(X)$ and $V(X)$.

5. Find the mgf of the random variable X of Problem 6.7. Using the mgf, find $E(X)$ and $V(X)$.

6. Suppose that the continuous random variable X has pdf
$$f(x) = \frac{1}{2}e^{-|x|}, \quad -\infty < x < \infty.$$
   (a) Obtain the mgf of X.
   (b) Using the mgf, find $E(X)$ and $V(X)$.

7. Use the mgf to show that if X and Y are independent random variables with distribution $N(\mu_x, \sigma_x^2)$ and $N(\mu_y, \sigma_y^2)$, respectively, then $Z = aX + bY$ is again normally distributed, where a and b are constants.

8. Suppose that the mgf of a random variable X is of the form
$$M_X(t) = (0.4e^t + 0.6)^8.$$
   (a) What is the mgf of the random variable $Y = 3X + 2$?
   (b) Evaluate $E(X)$.

(c) Can you check your answer to (b) by some other method? [Try to "recognize" $M_X(t)$.]

9. A number of resistances, $R_i, i = 1, 2, \ldots, n$, are put into a series arrangement in a circuit. Suppose that each resistance is normally distributed with $E(R_i) = 10$ ohms and $V(R_i) = 0.16$.

   (a) If $n = 5$, what is the probability that the resistance of the circuit exceeds 49 ohms?

   (b) How large should n be so that the probability that the total resistance exceeds 100 ohms is approximately 0.05?

10. In a circuit n resistances are hooked up into a series arrangement. Suppose that each resistance is uniformly distributed over $[0, 1]$ and suppose, furthermore, that all resistances are independent. Let R be the total resistance.

    (a) Find the mgf of R.

    (b) Using the mgf, obtain $E(R)$ and $V(R)$. Check your answers by direct computation.

11. If X has distribution x_n^2, using the mgf, show that $E(X) = n$ and $V(X) = 2n$.

12. Suppose that V, the velocity (cm/sec) of an object, has distribution $N(O, 4)$. If $K = mV^2/2$ ergs is the kinetic energy of the object (where m = mass), find the pdf of K. If $m = 10$ grams, evaluate $P(K \leq 3)$.

13. Suppose that the life length of an item is exponentially distributed with parameter 0.5. Assume that 10 such items are installed successively, so that the ith item is installed "immediately" after the $(i-1)$-item has failed. Let T_i be the time to failure of the i th item, $i = 1, 2, \ldots, 10$, always measured from the time of installation. Hence $S = T_1 + \cdots + T_{10}$, represents the total time of functioning of the 10 items. Assuming that the T_i's are independent, evaluate $P(S \geq 15.5)$.

14. Suppose that $X_1 \ldots, X_{80}$ are independent random variables, each having distribution $N(0, 1)$. Evaluate $P[X_1^2 + \cdots + X_{80}^2 > 77]$. [Hint: Use Theorem 9.2.]

15. Show that if $X_i, i = 1, 2, \ldots, k$, represents the number of successes in n_i repetitions of an experiment, where $P(\text{success}) = p$, for all i, then $X_1 + \cdots + X_k$ has a binomial distribution. (That is, the binomial distribution possesses the reproductive property.)

16. *(The Poisson and the multinomial distribution.)* Suppose that $X_i, i = 1, 2, \ldots, n$ are independently distributed· random variables having a Poisson distribution with parameters $a_i, i = 1, \ldots, n$. Let $X = \sum_{i=1}^{n} X_i$. Then the joint conditional probability distribution of $X_1, \ldots, X_n$ given $X = x$ is given by a multinomial distribution. That is,

$$P(X_1 = x_1, \ldots, X_n = x_n | X = x) = x!/(x_1! \ldots x_n!)(\alpha_1/\sum_{i=1\alpha_i}^{n})^{x_1 \cdots}(\alpha_n/\sum_{i=1\alpha_i}^{n})^{x_n}.$$

17. Obtain the mgf of a random variable having a geometric distribution. Does this distribution possess a reproductive property under addition?

18. If the random variable X has an mgf given by $M_x(t) = 3/(3-t)$, obtain the standard deviation of X.

19. Find the mgf of a random variable which is uniformly distributed over $(-1, 2)$.

20. A certain industrial process yields a large number of steel cylinders whose lengths are distributed normally with mean 3.25 inches and standard deviation 0.05 inch. If two such cylinders are chosen at random and placed end to end, what is the probability that their combined length is less than 6.60 inches?

Note: In evaluating $M'_X(t)$ at $t = 0$, an indeterminate form may arise. That is, $M'_X(0)$ may be of the form $0/0$. In such cases we must try to apply l'Hopital's rule. For example, if X is uniformly distributed over $[0,1]$, we easily find that $M_x(t) = (e^t - 1)/t$ and $M'_X(t) = (te^1 - e^t + 1)/t^2$. Hence at $t = 0, M'_X(t)$ is indeterminate. Applying l'Hopital's rule, we find that $lim_{t \to 0} M'_X(t) = lim_{t \to 0} te^t/2t = \frac{1}{2}$. This checks, since $M'_X(0) = E(X)$, which equals $\frac{1}{2}$ for the random variable described here.

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